Adaptive Metrics for Adaptive Samples

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Abstract

We generalize the local-feature size definition of adaptive sampling used in surface reconstruction to relate it to an alternative metric on Euclidean space. In the new metric, adaptive samples become uniform samples, making it simpler both to give adaptive sampling versions of homological inference results and to prove topological guarantees using the theory of critical points to distance functions.

1 From Surface Reconstruction to Homology Inference

To reconstruct a surface from a point set, one needs the sample to be sufficiently dense with respect to not just the local curvature of the surface, but also the distance to parts of the surface that are close in the embedding but far in geodesic distance. Otherwise, algorithms have no way of identifying which geometrically close sample points correspond to local neighborhoods in the surface. Adaptive sampling with respect to the so-called *local feature size* as introduced by Amenta and Bern [2] neatly characterizes such "good" samples and was then used in many later works on surface reconstruction with topological guarantees [7]. Such *adaptive samples* are in contrast to *uniform samples* for which a single parameter determines the density, usually driven by minimum of the local feature size and resulting in a much larger sample.

Later work on generalizations of surface reconstruction and homology inference related the topology of unions of balls centered at a sample $\widehat{X}$ near the unknown set X to the topology of X itself. A union of balls with a fixed radius can be viewed as a sublevel set of the distance function to $\widehat{X}$. If we have an adaptive sample, then we would like to scale the radii of the balls as well. However, if the sample is adaptive with respect to a local feature size defined as the distance to an unknown set L, another approximation $\widehat{L}$ near L is necessary. Indeed, one interpretation of some Voronoi-based surface reconstruction algorithms is that first an approximation $\widehat{L}$ to the medial axis L is computed from the Voronoi diagram of the sample $\widehat{X}$ of the unknown surface X.

We present a new perspective on adaptive samples. For *any* pair of disjoint, compact sets X and L, we define a metric on $\mathbb{R}^d \setminus L$ with the property that a uniform sample of X in the new metric corresponds to an adaptive sample in the Euclidean metric. This new metric can also be extended to an arbitrarily close Riemannian metric over the same domain. Our main motivation is to connect adaptive sampling theory to the critical point theory of distance functions used extensively to prove topological guarantees in topological data analysis [9, 4, 3]. That theory gives natural topological equivalences between sublevel sets of distance functions to compact sets in Riemannian metrics. Thus, we propose to use this new metric as the underlying ideal object and then relate it to a union of Euclidean balls constructed from approximations to X and L. Our metric can be viewed as a smoothed version of an adaptive metric used by Clarkson [5]. Our new formulation reveals connections with recent work on path planning [10, 1] and density-based distances [6].

2 Background

Let L and X be compact subsets of $\mathbb{R}^d$ with respect to the Euclidean metric. For $x, y \in \mathbb{R}^d$, define $\mathrm{Path}(x, y)$ to be the set of bounded piecewise-C_1 paths from x to y, parametrized by Euclidean arc-length. Similarly, $\mathrm{Path}(x, S) := \bigcup_{s \in S} \mathrm{Path}(x, s)$ denotes all paths from x to a set S.

For any compact set $L \subseteq \mathbb{R}^d$, define $f_L(\cdot) : \mathbb{R}^d \to \mathbb{R}$ by $f_L(x) := \min_{\ell \in L} \|x - \ell\|$. Define $\mathrm{d}^L(x, y) := \min_{\gamma \in \mathrm{Path}(x,y)} \int_\gamma \frac{dz}{f_L(z)}$. Note that d^L is a Riemannian metric on $\mathbb{R}^d \setminus L$. The length of a unit-speed path $\gamma : [0, a] \to \mathbb{R}^d$ is denoted as $|\gamma| := \int_\gamma dz = \int_0^a dt$.

For $y \in \mathbb{R}^d$, define $f_X^L(y) := \mathrm{d}^L(y, X) = \min_{x \in X} \mathrm{d}^L(y, x)$, and $\widehat{f_X^L}(y) := \min_{x \in X} \frac{\|y - x\|}{f_L(x)}$.

Note that $f_X^L(\cdot)$ is a distance function, while $\widehat{f_X^L}(\cdot)$ is not. The latter function can be interpreted as a first-order approximation of the former.

The two sets resulting from the level sets of these functions are defined below, with the goal being to approximate $A_X^L(\cdot)$ by $B_{\widehat{X}}^{\widehat{L}}(\cdot)$, where $\widehat{L}$ and $\widehat{X}$ are approximations of L and X respectively.

Definition 1 *For any compact set $X \subset \mathbb{R}^d \setminus L$, for some compact set $L \subset \mathbb{R}^d$, the* α-offsets *with respect to*

d^L *are*

$$A_X^L(\alpha) := \{x \in \mathbb{R}^d \mid f_X^L(x) \leq \alpha\}.$$

Note the distance function $f_L(\cdot)$ can be transformed into an arbitrarily close smooth function $\tilde{f}_L(\cdot)$ [8], yielding a Riemannian metric $\tilde{\mathrm{d}}_L$ defined in an identical manner as d_L. From this, one has corresponding α-offsets $\tilde{A}_X^L(\alpha)$ that are arbitrarily close to $A_L^X(\alpha)$.

Definition 2 *For any compact set $X \subset \mathbb{R}^d \setminus L$, for some compact set $L \subset \mathbb{R}^d$, the* approximate α-offsets *with respect to* d^L *are*

$$B_X^L(\alpha) := (\widehat{f_X^L})^{-1}[0, \alpha] = \bigcup_{x \in X} \mathrm{ball}(x, \alpha f_L(x)).$$

A useful property of $f_X^L(\cdot)$ is that it a 1-Lipschitz function. In general, a function f between two metric spaces (X, d_X) and (Y, d_Y) is said to be k-Lipschitz if for all $x, y \in X$, $\mathrm{d}_Y(f(x), f(y)) \leq k\mathrm{d}_X(x, y)$.

Lemma 3 *$f_X^L(\cdot)$ is a 1-Lipschitz function from $(\mathbb{R}^d, \mathrm{d}^L)$ to $\mathbb{R}$.*

Proof. Fix any $a, b \in \mathbb{R}^d$. There exists $x \in X$ and $\gamma_1 \in \mathrm{Path}(a, x)$ such that $f_X^L(a) = \int_{\gamma_1} \frac{dz}{f_L(z)}$. Likewise, there exists $\gamma_2 \in \mathrm{Path}(a, b)$ such that $\mathrm{d}^L(a, b) = \int_{\gamma_2} \frac{dz}{f_L(z)}$.

This implies $\gamma_1 + \gamma_2 \in \mathrm{Path}(b, X)$, where $+$ in this case is the concatenation of paths in the usual sense.

Thus $f_X^L(b) \leq \int_{\gamma_1+\gamma_2} \frac{dz}{f_L(z)} \leq f_X^L(a) + \mathrm{d}^L(a, b)$. By symmetry of a and b, we obtain the other bound and we are done. $\square$

We can extend $f_X^L(\cdot)$, a function measuring the distance from a point to a set, to the resulting Hausdorff distance, which is a metric between compact sets. This metric is useful for stating bounds on the quality, or uniformity, of a sample near a set.

Definition 4 *The* Hausdorff distance *between two compact sets $A, B \in (\mathbb{R}^d, \mathrm{d}^L)$ is defined as*

$$\mathrm{d}_H^L(A, B) = \max\{\min_{a \in A} f_B^L(a), \min_{b \in B} f_A^L(b)\}$$

or equivalently,

$$\mathrm{d}_H^L(A, B) = \min\{r \mid A \subseteq B_L^r \text{ and } B \subseteq A_L^r\}.$$

Using an assumption on the Hausdorff distance between a compact set and a sample of it, Lemma 5 shows their α-offsets can be included within each other at particular scales.

Lemma 5 *Consider $\widehat{X}, X \subseteq \mathbb{R}^d \setminus L$ be such that $\mathrm{d}_H^L(\widehat{X}, X) \leq \delta$. Then for all $\alpha \geq 0$, $A_X^L(\alpha) \subseteq A_{\widehat{X}}^L(\alpha+\delta)$ and $A_{\widehat{X}}^L(\alpha) \subseteq A_X^L(\alpha+\delta)$.*

Proof. Fix $y \in A_X^L(\alpha)$. By definition $f_X^L(y) \leq \alpha$, which implies that there exists $x \in X$ such that $\mathrm{d}^L(x, y) \leq \alpha$. $\mathrm{d}_H^L(\widehat{X}, X) \leq \delta$ which implies that for all $x \in X$, $f_{\widehat{X}}^L(x) \leq \delta$. Now by Lemma 3, $f_{\widehat{X}}^L(y) \leq f_{\widehat{X}}^L(x) + \mathrm{d}^L(x, y) \leq \delta + \alpha$, implying $y \in A_{\widehat{X}}^L(\alpha+\delta)$. By a symmetric argument, the other statement holds. $\square$

The following is the definition of an adaptive sample we use, known as an ε-sample.

Definition 6 *Given compact set $L \subset \mathbb{R}^d$ and compact sets $X, \widehat{X} \subset \mathbb{R}^d \setminus L$ such that $\widehat{X} \subseteq X$, we say that $\widehat{X}$ is an* ε-sample *of X, for $\varepsilon \in [0, 1)$, if for all $x \in X$, there exists $p \in \widehat{X}$ such that $\|x - p\| \leq \varepsilon f_L(x)$.*

This definition is closely related to that of the approximate α-offsets, because if $\widehat{X}$ is an ε-sample of X, then for all $x \in X$, $\mathrm{ball}(x, \varepsilon f_L(x)) \cap \widehat{X} \neq \emptyset$.

3 Adaptive Sampling

In this section, we prove that a uniform sample in the induced metric corresponds to an adaptive sample in the Euclidean metric and vice versa. The key to this proof is the following lemma about the relationship between the two metrics when just considering two points. This lemma will also be used for the more elaborate interleaving results of Section 4.

Lemma 7 *Let $L \subset \mathbb{R}^d$ be a compact set and let $a, b \in \mathbb{R}^d \setminus L$. Then, the following two statements hold for all $\delta \in [0, 1)$.*

(i) If $\mathrm{d}^L(a, b) \leq \delta$ then $\frac{\|a-b\|}{f_L(a)} \leq \frac{\delta}{1-\delta}$.

(ii) If $\frac{\|a-b\|}{f_L(a)} \leq \delta$ then $\mathrm{d}^L(a, b) \leq \frac{\delta}{1-\delta}$.

Proof. To prove (i), we assume $\mathrm{d}^L(a, b) \leq \delta$. Let γ be the path in $\mathrm{Path}(a, b)$ such that $\mathrm{d}^L(a, b) = \int_\gamma \frac{dz}{f_L(z)} < \delta$. Then we have the following inequalities following from the Lipschitz property of f_L.

$$\begin{aligned}
|\gamma| = \int_\gamma dz &= (f_L(a) + |\gamma|) \int_\gamma \frac{dz}{f_L(a) + |\gamma|} \\
&\leq (f_L(a) + |\gamma|) \int_\gamma \frac{dz}{f_L(z)} \\
&\leq (f_L(x) + |\gamma|)\delta
\end{aligned}$$

It follows that $|\gamma| \leq \frac{\delta}{1-\delta} f_L(x)$. Because $\|a - b\|$ is the length of the shortest path between a and b in the Euclidean metric, we conclude $\|a - b\| \leq |\gamma| \leq \frac{\delta}{1-\delta} f_L(x)$.

Next we prove (ii). Assume $\frac{\|a-b\|}{f_L(a)} \leq \delta$. For all points z in the straight line segment $\overline{ab}$,

$$f_L(z) \geq f_L(a) - \|a - z\| \geq f_L(a) - \|a - b\| \geq (1 - \delta) f_L(a).$$

This implies the following inequality.

$$\begin{aligned}
\mathrm{d}^L(a,b) &= \inf_{\gamma\in \mathrm{Path}(a,b)} \int_\gamma \frac{dz}{f_L(z)} \\
&\le \int_{\overline{ab}} \frac{dz}{f_L(z)} \\
&\le \frac{1}{(1-\delta)f_L(a)} \int_{\overline{ab}} dz \\
&= \frac{\|a-b\|}{(1-\delta)f_L(a)} \\
&\le \frac{\delta}{1-\delta}.
\end{aligned}$$

□

We can now state the main theorem relating adaptive samples in the Euclidean metric to uniform samples in the metric induced by a set L.

Theorem 8 *Let L and X be compact sets, let $\widehat{X} \subseteq X$ be a sample, and let $\varepsilon \in [0,1)$ be a constant. If $\widehat{X}$ is an ε-sample of X with respect to the distance to L, then $\mathrm{d}^L_H(X,\widehat{X}) \le \frac{\varepsilon}{1-\varepsilon}$. Also, if $\mathrm{d}^L_H(X,\widehat{X}) \le \varepsilon < \frac{1}{2}$, then $\widehat{X}$ is an $\frac{\varepsilon}{1-\varepsilon}$-sample of X with respect to the distance to L.*

Proof. Given $x \in X$, there exists $p \in \widehat{X}$ such that $\|x-p\| \le \varepsilon f_L(x)$. By Lemma 7, $\mathrm{d}^L(x,p) \le \frac{\varepsilon}{1-\varepsilon}$, so for all $x \in X$, $f^L_{\widehat{X}}(x) \le \frac{\varepsilon}{1-\varepsilon}$. As $\widehat{X} \subseteq X$, this proves $\mathrm{d}^L_H(\widehat{X},X) \le \frac{\varepsilon}{1-\varepsilon}$.

$\mathrm{d}^L_H(\widehat{X},X) \le \varepsilon < \frac{1}{2}$ implies that for all $x \in X$, $f^L_{\widehat{X}}(x) \le \varepsilon$, thus there exists $p \in \widehat{X}$ such that $\mathrm{d}^L(x,p) \le \varepsilon$, and thus by Lemma 7 $\|x-p\| \le \frac{\varepsilon}{1-\varepsilon} f_L(x)$. Since $\varepsilon < \frac{1}{2}$, then $\frac{\varepsilon}{1-\varepsilon} < 1$, so $\widehat{X}$ is an $\frac{\varepsilon}{1-\varepsilon}$-sample of X. □

4 Interleaving

A *filtration* is a nested family of sets. In this paper, we consider filtrations F parameterized by a real number $\alpha \ge 0$ so that $F(\alpha) \subset \mathbb{R}^d$ and whenever $\alpha < \beta$ we have $F(\alpha) \subseteq F(\beta)$. Often, our filtrations are *sublevel filtrations* of a real valued function $f : \mathbb{R}^d \to \mathbb{R}$. The sublevel filtration F corresponding to the function f is the defined as

$$F(\alpha) := \{x \in \mathbb{R}^d \mid f(x) \le \alpha\}.$$

Definition 9 *A pair of filtrations (F,G)* is (h_1,h_2)-interleaved *in an interval (s,t) if $F(r) \subseteq G(h_1(r))$ whenever $r, h_1(r) \in (s,t)$ and $G(r) \subseteq F(h_2(r))$ whenever $r, h_2(r) \in (s,t)$. We require that the functions h_1, h_2 be nondecreasing in (s,t).*

The following lemma gives us an easy iterative way to combine pairs of interleavings.

Lemma 10 *If (F,G) is (h_1,h_2)-interleaved in (s_1,t_1), and (G,H) is (h_3,h_4)-interleaved in (s_2,t_2), then (F,H) is $(h_3 \circ h_1, h_2 \circ h_4)$-interleaved in (s_3,t_3), where $s_3 = \max\{s1, s2\}$ and $t_3 = \min\{t_1,t_2\}$.*

Proof. If $r, h_3(h_1(r)) \in (s_3,t_3)$, then we have $F(r) \subseteq G(h_1(r)) \subseteq H(h_3(h_1(r)))$. Similarly, if $r, h_2(h_4(r)) \in (s_3,t_3)$, then $H(r) \subseteq G(h_4(r)) \subseteq F(h_2(h_4(r)))$. □

4.1 Approximating X with $\widehat{X}$

Ultimately, the goal is to relate A^L_X, the offsets in the induced metric, to $B^{\widehat{L}}_{\widehat{X}}$, the approximate offsets computed from approximations (or samples) to both X and L. This relationship will be given by an interleaving that is built up from an interleaving for each approximation step. For each of the following lemmas, let $L, \widehat{L} \subset \mathbb{R}^d$ and $X, \widehat{X} \subset \mathbb{R}^d \setminus (L \cup \widehat{L})$ be compact sets.

Lemma 11 *If $\mathrm{d}^L_H(\widehat{X},X) \le \varepsilon$, then $(A^L_X, A^L_{\widehat{X}})$ is (h_1,h_1)-interleaved in $(0,\infty)$, where $h_1(r) = r+\varepsilon$.*

Proof. This lemma is a reinterpretation of Lemma 5 in the interleaving notation. □

4.2 Approximating the Induced Metric

It is much easier to use a union of Euclidean balls to model the sublevel sets of the distance function f^L_X. Below, we show that this is a reasonable approximation. The following results may also be viewed as a strengthening of the adaptive sampling result of the previous section (Theorem 8).

Lemma 12 *Given compact set $L \subset \mathbb{R}^d$, and compact set $X \subset \mathbb{R}^d \setminus L$, for $r \in [0,1)$, $A^L_X(r) \subseteq B^L_X(\frac{r}{1-r})$, and for $r \in [0,\frac{1}{2})$, $B^L_X(r) \subseteq A^L_X(\frac{r}{1-r})$.*

Proof. Take $y \in A^L_X(r)$ so that $f^L_X(y) \le r$. Thus there exists $x \in X$ such that $\mathrm{d}^L(x,y) \le r$. By Lemma 7, this implies that $\|x-y\| \le \frac{r}{1-r} f_L(x)$, which implies that $y \in B^L_X(\frac{r}{1-r})$.

Consider $y \in B^L_X(r)$. Thus $y \in \mathrm{ball}(x, rf_L(x))$, for some $x \in X$, so $\|x-y\| \le rf_L(x)$. Applying Lemma 7, we have then have that $\mathrm{d}^L(x,y) \le \frac{r}{1-r}$, and as $f^L_X(y) \le \mathrm{d}^L(x,y)$, $y \in A^L_X(\frac{r}{1-r})$. □

Corollary 13 *The pair $(A^L_{\widehat{X}}, B^L_{\widehat{X}})$ are (h_2,h_2)-interleaved in $(0,1)$, where $h_2(r) = \frac{r}{1-r}$.*

Proof. This follows from combining the results of Lemma 12 into the interleaving notation.

□

4.3 Approximating L with $\widehat{L}$

Usually, the set L is unknown at the start and must be estimated from the input. For example in the case that L is the medial axis of X, there are several known techniques for approximating L by, for example, taking some vertices of the Voronoi diagram [2, 7]. We would like to give some sampling conditions that guarantee that allow us to replace L with an approximation $\widehat{L}$. Interestingly, the sampling conditions for $\widehat{X}$ are dual to those used for $\widehat{L}$. That is, we require $\mathrm{d}_H^{\widehat{X}}(L, \widehat{L}) \le \varepsilon$, or, in other words, $\widehat{L}$ must be an adaptive sample with respect to the distance to $\widehat{X}$.

Lemma 14 *If $\mathrm{d}_H^{\widehat{X}}(L, \widehat{L}) \le \delta < 1$, then $(B^L_{\widehat{X}}, B^{\widehat{L}}_{\widehat{X}})$ is (h_3, h_3)-interleaved in $(0, \infty)$, where $h_3(r) = \frac{r}{1-\delta}$.*

Proof. Fix any $x \in B^L_{\widehat{X}}(r)$. There is a point $p \in \widehat{X}$ such that $\frac{\|x-p\|}{f_L(p)} \le r$. Moreover, there is a nearest point $z \in \widehat{L}$ to x, and so $f_{\widehat{L}}(p) = \|p - z\|$. Lemma 7 and the assumption that $\mathrm{d}_H^{\widehat{X}}(L, \widehat{L}) \le \delta$ implies that there exists $y \in L$ such that

$$\|y - z\| \le \frac{\delta}{1-\delta} f_{\widehat{X}}(z). \qquad (1)$$

The definitions imply the following.

$$f_{\widehat{X}}(z) = \min_{q \in \widehat{X}} \|z - q\| \le \|z - p\| = f_{\widehat{L}}(p). \qquad (2)$$

So, we can bound $f_L(p)$ in terms of $f_{\widehat{L}}(p)$ as follows.

$$\begin{aligned} f_L(p) &\le \|y - p\| && [y \in L] \\ &\le \|y - z\| + \|z - p\| && [\text{triangle inequality}] \\ &\le \frac{1}{1-\delta} f_{\widehat{L}}(p) && [\text{by (1) and (2)}] \end{aligned}$$

So,

$$\frac{\|x-p\|}{f_{\widehat{L}}(p)} \le \frac{\|x-p\|}{(1-\delta) f_L(p)} \le \frac{r}{1-\delta} = h_3(r).$$

Therefore, $x \in B^{\widehat{L}}_{\widehat{X}}(h_3(r))$ and so we conclude that $B^L_{\widehat{X}}(r) \subseteq B^{\widehat{L}}_{\widehat{X}}(h_3(r))$. The proof is symmetric to show that $B^{\widehat{L}}_{\widehat{X}}(r) \subseteq B^L_{\widehat{X}}(h_3(r))$ □

4.4 Putting it all together

We can now combine the interleavings established in Corollary 13, and Lemmas 11 & 14, using Lemma 10.

Theorem 15 *Let $L, \widehat{L} \subset \mathbb{R}^d$ and $X, \widehat{X} \subset \mathbb{R}^d \setminus (L \cup \widehat{L})$ be compact sets. If $\mathrm{d}_H^{\widehat{X}}(L, \widehat{L}) \le \delta < 1$ and $\mathrm{d}_H^L(\widehat{X}, X) \le \varepsilon < 1$, then $(A^L_X, B^{\widehat{L}}_{\widehat{X}})$ are (h_4, h_5)-interleaved in $(0, 1)$, where $h_4(r) = \frac{r+\varepsilon}{(1-r-\varepsilon)(1-\delta)}$ and $h_5(r) = \frac{r}{1-\delta-r} + \varepsilon$.*

Proof. By Lemma 10 along with the interleavings from Lemmas 11, 13, $(A^L_X, B^L_{\widehat{X}})$ is $(h_2 \circ h_1, h_1 \circ h_2)$-interleaved in $(0, 1)$. Combining this interleaving with the one resulting from Lemma 14 we get that $(A^L_X, B^{\widehat{L}}_{\widehat{X}})$ is $(h_3 \circ h_2 \circ h_1, h_1 \circ h_2 \circ h_3)$ interleaved in $(0, 1)$. Now we must compute $h_3 \circ h_2 \circ h_1$ and $h_1 \circ h_2 \circ h_3$.

$$\begin{aligned} (h_3 \circ h_2 \circ h_1)(r) = (h_3 \circ h_2)(r + \delta) &= h_3\left(\frac{r+\delta}{1-r-\delta}\right) \\ &= \frac{r+\delta}{(1-r-\delta)(1-\varepsilon)} \\ (h_1 \circ h_2 \circ h_3)(r) = (h_1 \circ h_2)\left(\frac{r}{1-\varepsilon}\right) &= h_1\left(\frac{r}{(1-\varepsilon)(1-\frac{r}{1-\varepsilon})}\right) \\ &= h_1\left(\frac{r}{1-\varepsilon-r}\right) \\ &= \frac{r}{1-\varepsilon-r} + \delta \end{aligned}$$

So we have that $h_4(r) = \frac{r+\delta}{(1-r-\delta)(1-\varepsilon)}$ and $h_5(r) = \frac{r}{1-\varepsilon-r} + \delta$. □

5 Conclusion

In our paper, we present results based on an alternative metric in Euclidean space that connect adaptive sampling and uniform sampling. With a metric comes a distance function with which one can apply classical results from critical point theory to infer topological properties of the underlying space, thus providing a connection between surface reconstruction (adaptive sampling) and homology inference (uniform sampling). Since one does not know the exact compact set X being reconstructed, nor the reference set L on which the adaptive sample is based, approximations $\widehat{X}$ and $\widehat{L}$ are needed.

We show in Theorem 8 that there is a precise relationship between samples that are uniformly taken with respect to d^L at some scale, to those same samples being adaptive in the Euclidean metric. In our main result, Theorem 15, we show that we can interleave the sublevel sets of our distance function under this alternative metric with the metric balls resulting from our approximation of the metric, assuming that both $\widehat{X}$ and $\widehat{L}$ are uniformly well-sampled with respect to the Hausdorff distance of d^L and $\mathrm{d}^{\widehat{X}}$. Using all approximations, albeit well-chosen ones, one can infer the behavior of the defined metric as well as the sublevel sets of it with respect to X.

There is a natural next step building off of this research that broadens its scope, background, and further applications. With the aforementioned critical point theory, these interleavings could be extended to homological guarantees about the compact set X in question.

An application of such a result could be that obstacle avoidance results could be reframed as obstacle exploration.

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